Nonequilibrium statistical physics with fictitious time

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Problems in nonequilibrium statistical physics are characterized by the absence of a fluctuation dissipation theorem. The usual analytic route for treating these vast class of problems is to use response fields in addition to the real fields that are pertinent to a given problem. This line of argument was introduced by Martin, Siggia, and Rose. We show that instead of using the response field, one can, following the stochastic quantization of Parisi and Wu, introduce a fictitious time. In this extra dimension a fluctuation dissipation theorem is built in and provides a different outlook to problems in nonequilibrium statistical physics.

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Study of dynamics near equilibrium is facilitated by the existence of the fluctuation dissipation theorem which relates the correlation function to the response function. One of the primary difficulties in the study of dynamics far from equilibrium is the absence of a fluctuation dissipation theorem leading to independent diagrammatic expansions for the response function and the correlation function. The difficulty stems from the fact that the equilibrium distribution is not known and the only way in which we can do an averaging is over noise. The spectacular success in the last few decades in the study of dynamics has been in the near-equilibrium dynamics near second-order phase transitions. Success in this case implies a close and detailed correspondence between theory and experiment in a variety of systems [1–6]. The dynamics in this case has typically been that of an n-component field $\phi_\alpha(\alpha=1,2,\ldots,n)$ satisfying an equation of motion (model A, model B, etc.),

$$\frac{\partial\phi_\alpha}{\partial t} = -\Gamma\frac{\delta F}{\delta\phi_\alpha} + \eta_\alpha, \tag{1}$$

where F is a free-energy functional (most often the Ginzburg-Landau free energy), with the Gaussian white noise $\eta(\vec{r},t)$ satisfying

$$\langle\eta_\alpha(\vec{r}_1,t_1)\,\eta_\beta(\vec{r}_2,t_2)\rangle = 2\Gamma kT\delta_{\alpha\beta}\delta(\vec{r}_1-\vec{r}_2)\,\delta(t_1-t_2). \tag{2}$$

It is straightforward to check by using to the associated Fokker-Planck equation that $\exp(-F/kT)$ is indeed the equilibrium distribution. Since the dynamics is always close to equilibrium, it is the averaging with the distribution $\exp(-F/kT)$ which one has in mind. There have been more complicated equations of motion with the structure

$$\frac{\partial\phi_\alpha}{\partial t} = V_\alpha([\phi_\beta]) - \Gamma\frac{\delta F}{\delta\phi_\alpha} + \eta_\alpha \tag{3}$$

(e.g., models E,F,G,H,J of dynamic critical phenomena), but the equilibrium distribution has been maintained because $\frac{\partial\phi_\alpha}{\partial t}=V_\alpha([\phi_\beta])$ keeps the free-energy F constant in time (in fact that is the motivation behind the construction of V_α).

Models of nonequilibrium statistical mechanics do not evolve towards an equilibrium distribution. Consequently, the averaging involved in the construction of correlation function has to be done explicitly over the noise inherent in the statistical dynamics [the noise written explicitly in Eqs. (1) and (2)]. One of the most-studied models in this category is the Kardar-Parisi-Zhang (KPZ) equation, where the dynamics (physically the growth of a surface by deposition of atoms on a substrate) is given by

$$\frac{\partial\phi(\vec{r},t)}{\partial t} = \nu\nabla^2\phi - \frac{\lambda}{2}(\vec{\nabla}\phi)^2 + \eta, \tag{4}$$

where $\langle\eta(\vec{r}_1,t_1)\,\eta(\vec{r}_2,t_2)\rangle = 2D_0\delta(\vec{r}_1-\vec{r}_2)\,\delta(t_1-t_2)$.

The KPZ equation models a growing interface which develops from a random deposition of atoms on a substrate of dimensions D. The randomness in the deposition process brings in the noise term in the time evolution of the height of the interface. The deterministic part of the evolution coming from the deposition and evaporation of atoms is governed by the chemical potential μ, which is determined by the total height gradient and its derivatives. Clearly a linear term in the gradient is not acceptable since the evolution cannot depend on the sign of the gradient. Thus the linear term in μ has to be the second derivative of the height and the nonlinear term is the square of the gradient. Without the nonlinear term, one has the Edwards Wilkinson model [8] of growth and with it the KPZ model. One of the primary uses in the models of growth is the question about the roughness of the interface. This is expressed in terms of the correlation function $\langle\phi(\vec{x}+\vec{r},t)\phi(\vec{x},t)\rangle$, which scales as $r^{2\alpha}$ for large r. If $\alpha>0$, then the height fluctuations are coupled at long distances as the surface is rough. If $\alpha<0$, the fluctuations decay and the surface is smooth. The EW model has $\alpha=(2-D)/2$, leading to a rough interface for $D\leqslant 2$. For the KPZ model, $\alpha>0$ and the surface is always rough for $D<D_c$, where D_c an upper critical dimension. For $D>D_c$, the surface is smooth for small values of the coupling constant. For $D>2$, there is a strong and weak coupling regime and the existence of D_c is related to the strong coupling phase. The value of D_c is still controversial [9–12].

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Now, Eq. (4) is not of the form of Eq. (1) $[\nu\nabla^2\phi-\frac{\lambda}{2}(\vec{\nabla}\phi)^2$ cannot be written in the form $\delta F/\delta\phi]$. It is of the form of Eq. (3), with $F=(\nu/2)\int d^Dx(\vec{\nabla}\phi)^2$, but, identifying $V([\phi])$ as $-(\lambda/2)(\vec{\nabla}\phi)^2$, it can be shown that F is preserved only in $D=1$ (D is the substrate dimension).

The field theoretic technique of handling such problems was first explained by Martin, Siggia, and Rose [13] and perfected over the years by Janssen [14], Bausch *et al.* [15], De Dominicis and Peliti [16], Frey and Täuber [17], Plischke *et al.* [18], and a host of other researchers. One begins by writing the "partition function"

$$Z=\int\mathcal{D}\phi\int\mathcal{D}\eta e^{-1/2D_0\int d^Dxdt\,\eta(\vec{r},t)^2}$$

$$\delta\left(\frac{\partial\phi}{\partial t}-\nu\nabla^2\phi-\frac{\lambda}{2}(\vec{\nabla}\phi)^2-\eta(\vec{r},t)\right). \qquad (5)$$

Integrating over η, $Z=\int\mathcal{D}\phi\exp-S(\phi)$, with the action

$$S(\phi)=\frac{1}{D_0}\int d^D\vec{r}dt\left[\frac{\partial\phi}{\partial t}-\nu\nabla^2\phi-\frac{\lambda}{2}(\vec{\nabla}\phi)^2\right]^2, \qquad (6)$$

as given by Zee [21].

It is with the action that all correlations of the form $\langle\phi(\vec{r}_1,t_1)\phi(\vec{r}_2,t_2)\rangle$ have to be determined. The response functions are introduced by writing Z in terms of an auxiliary field, so that

$$Z=\int\mathcal{D}[\phi]\mathcal{D}[\tilde{\phi}]e^{-S(\phi,\tilde{\phi})}, \qquad (7)$$

where the new action is given by

$$S(\phi,\tilde{\phi})=\int dt\int d^D\vec{r}\left[\tilde{\phi}(\vec{r},t)\left[\dot{\phi}-\nu\nabla^2\phi-\frac{\lambda}{2}(\vec{\nabla}\phi)^2\right]-D\tilde{\phi}^2\right]. \qquad (8)$$

The way to handle this case of nonequilibrium statistical mechanics is delineated clearly in Frey and Täuber [17].

Our proposal here is to use the action of Eq. (6) and exploit the principle of stochastic quantization to set up an alternative approach to the study of this class of problems. Stochastic quantization is a method of quantization proposed by Parisi and Wu [19,20] based on stochastic Langevin dynamics of a physical system in a fifth time τ. They showed that, at the perturbative level, the usual quantum field theory was recovered in the limit $\tau\to\infty$ of this dynamics. Euclidean quantum field theory correlation functions for a field ϕ corresponding to an action $S(\phi)$ are given by

$$\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_l)|0\rangle=\frac{\int\mathcal{D}[\phi]\phi(x_1)\cdots\phi(x_l)e^{-S[\phi]}}{\int\mathcal{D}[\phi]e^{-S[\phi]}}.$$

Parisi and Wu proposed the following alternative method:

(a) Introduce an extra fictitious "time" τ in addition to the four space-time X^μ and postulate a Langevin dynamics

$$\frac{\partial\phi(x,t,\tau)}{\partial\tau}=-\frac{\delta S}{\delta\phi}+f(x,t,\tau), \qquad (9)$$

where f is a Gaussian random variable with $\langle f(x,t,\tau)f(x',t'\tau')\rangle=2\delta(x-x')\delta(t-t')\delta(\tau-\tau')$.

(b) Evaluate the stochastic average of the fields $\phi(x,t,\tau)$ satisfying Eq. (9) i.e., evaluate $\langle\phi(x_1,t_1,\tau_1)\phi(x_2,t_2,\tau_2)\cdots\phi(x_l,t_l,\tau_l)\rangle_\eta$.

(c) Set $\tau_1=\tau_2=\cdots=\tau_l=\tau$ and take the limit $\tau\to\infty$. Then, one has

$$\lim_{\tau\to\infty}\langle\phi(x_1,t_1,\tau)\phi(x_2,t_2,\tau)\cdots\phi(x_l,t_l,\tau)\rangle_\eta$$

$$=\frac{\int\mathcal{D}[\phi]\phi(x_1,t_1)\phi(x_2,t_2)\cdots\phi(x_l,t_l)e^{-S[\phi]}}{\int\mathcal{D}[\phi]e^{-S[\phi]}}. \qquad (10)$$

What we will demonstrate is that, since the dynamics of Eq. (9) requires only the calculation of correlation functions (response functions in this dynamics are related to the correlation functions because of the fluctuation dissipation theorem), we can obtain in a straightforward fashion the scaling laws in real space and time from the solution in the τ space. We first demonstrate that the method works in the standard situations.

Let us start with a toy model as the simplest example of a dynamical system that never reaches equilibrium, namely

$$\dot{X}(t)=\eta(t)\text{ with }\langle\eta(t)\eta(t')\rangle=2\delta(t-t'), \qquad (11)$$

and write the action of Eq. (6) as

$$S[X(t)]=\frac{1}{4}\int\left(\frac{dX}{dt}\right)^2dt. \qquad (12)$$

The Langevin equation in the fifth time τ is now written as

$$\frac{\partial X}{\partial\tau}=\frac{1}{2}\frac{\partial^2X}{\partial t^2}+f(t,\tau), \qquad (13)$$

with $\langle f(t,\tau)f(t',\tau')\rangle=\delta(t-t')\delta(\tau-\tau')$. The solution of Eq. (13) is

$$X(t,\tau)=\int_0^t dt'\int_0^\tau d\tau'\frac{1}{\sqrt{\tau'}}e^{-(t'^2/2\tau')}f(t',\tau'). \qquad (14)$$

The correlation function $C(w,\tau,\tau')=\langle X(w,\tau)X(w,\tau')\rangle$ and response function $R(w,\tau,\tau')=2\langle X(w,\tau)f(w,\tau')\rangle$ satisfy the fluctuation dissipation theorem in fictitious time as

$$\frac{\partial}{\partial\tau'}C(\tau,\tau')=R(\tau,\tau'). \qquad (15)$$

With the help of Eq. (14), we calculate the correlation function in the $\tau\to\infty$ limit as

$$\frac{\langle X(t_1,\tau)X(t_2,\tau)\rangle}{\langle X(t_1,\tau)^2\rangle^{1/2}\langle X(t_2,\tau)^2\rangle^{1/2}}=\sqrt{\frac{t_2}{t_1}}\text{ if }t_2<t_1, \qquad (16)$$

which is the standard result.

We now turn to the relaxational dynamics of a simple free-scalar field $\phi(x,t)$, given by

$$\frac{\partial\phi(x,t)}{\partial t} = \nabla^2\phi(x,t) - m^2\phi(x,t) + \eta(x,t), \tag{17}$$

with $\langle \eta(x,t)\eta(x',t')\rangle = 2\delta(t-t')$. The action in momentum-frequency space is written as

$$S = \frac{1}{4}\int \frac{d^D\vec{k}}{(2\pi)^D}\int \frac{dw}{(2\pi)}|[-iw + (k^2+m^2)]\phi(\vec{k},w)|^2. \tag{18}$$

The Langevin equation in fifth time τ is now written as

$$\frac{\partial\phi(\vec{l},\omega,\tau)}{\partial\tau} = -\frac{1}{2}[\omega^2 + (l^2+m^2)]\phi(\vec{l},\omega,\tau) + f(\vec{l},\omega,\tau), \tag{19}$$

with $\langle f(\vec{l},\omega,\tau)f(\vec{l},\omega',\tau')\rangle = \delta(\vec{l}+\vec{l}')\delta(\omega+\omega')\delta(\tau-\tau')$. Clearly in the τ variable the fluctuation dissipation theorem is satisfied.

We now calculate the correlation function by setting $\tau_1 = \tau_2 = \tau$ and, in the $\tau\to\infty$, we obtain

$$\langle\phi(x_1,t_1)\phi(x_2,t_2)\rangle \sim \frac{1}{(t_1-t_2)^{(D/2)-1}} \times e^{-m^2(t_1-t_2)}e^{-[(x_1-x_2)^2]/[4(t_1-t_2)]}, \tag{20}$$

where $\phi(x,t)$ is the solution of Eq. (19). For $m=0$ and for large $(t_1-t_2)=t$, $\langle\phi(x_1,t_1)\phi(x_2,t_2)\rangle \sim t^{-\delta}$ with $\delta = \frac{D}{2}-1$.

To obtain the response function of the original theory, we need to consider $\langle X(t,\tau)f(t',\tau')\rangle$, then set $\tau=\tau'$, and integrate with respect to the real time.We can obtain the response function in both the random walk and free-scalar field case easily in this way.

We now return to the KPZ equation and write the action of Eq. (6) in momentum-frequency space as

$$S = \frac{1}{4D_0}\int \frac{d^Dk}{(2\pi)^D}\frac{dw}{2\pi}\Bigg\{(w^2+\nu^2k^4)\phi(\vec{k},w)\phi(-\vec{k},-w) + \frac{\lambda}{2}\sum_{\vec{q},w'}(-iw+\nu k^2)\vec{q}\cdot(\vec{k}+\vec{q})\phi(\vec{k},w)\phi(\vec{q},w')\phi(-\vec{k}-\vec{q},-w-w') - \frac{\lambda}{2}\sum_{\vec{q},w'}(iw+\nu k^2)\vec{q}\cdot(\vec{k}-\vec{p})\phi(\vec{q},w')\phi(-\vec{k}-w)\phi(\vec{k}-\vec{q},w-w') - \frac{\lambda^2}{4}\sum_{\vec{p},\vec{q},w',w''}[\vec{p}\cdot(\vec{k}-\vec{p})][\vec{q}\cdot(\vec{k}+\vec{q})]\phi(\vec{p},w')\phi(\vec{k}-\vec{p},w - w')\phi(-\vec{k}-\vec{q},-w-w'')\phi(\vec{q},w'')\Bigg\}, \tag{21}$$

in accordance with standard results [21]. The Langevin equation in the fifth time τ is now written as

$$\frac{\partial\phi(\vec{l},\Omega,\tau)}{\partial\tau} = \frac{\delta S}{\delta\phi(-\vec{l},-\Omega,\tau)} + f, \tag{22}$$

with

$$\langle f(\vec{l},\Omega,\tau)f(\vec{l}',\Omega',\tau')\rangle = 2\delta(\vec{l}+\vec{l}',\Omega+\Omega',\tau-\tau').$$

After straightforward algebra, Eq. (22) acquires the form

$$\frac{\partial\phi(l,\Omega,\tau)}{\partial\tau} = -\frac{(\Omega^2+\nu^2l^4)}{2D_0}\phi(l,\Omega,\tau) - \frac{\lambda}{4D_0}\sum_{\vec{k},w}[2(-iw + \nu k^2)\vec{l}(\vec{l}-\vec{k}) - (i\Omega+\nu l^2)\vec{k}\cdot(\vec{l}-\vec{k})]\phi(\vec{k},w,\tau)\phi(\vec{l}-\vec{k},\Omega-w,\tau) + \frac{\lambda^2}{4D_0}\sum_{\vec{k},\vec{q},w,w'}[\vec{l}\cdot(\vec{l}-\vec{k})][\vec{q}\cdot(\vec{k}-\vec{q})]\phi(\vec{q},w',\tau)\phi(\vec{k}-\vec{q},w-w',\tau')\phi(\vec{l}-\vec{k},\Omega-w,\tau') + f(l,\Omega,\tau). \tag{23}$$

For the free-field theory (Edwards Wilkinson model) $\lambda=0$ and we get the response function of the system as

$$G_0 = \left(-iw_\tau + \frac{w^2+\nu^2k^4}{2D_0}\right)^{-1}, \tag{24}$$

where w_τ is the Fourier transform variable corresponding to the fictitious time τ. The correlation function is

$$C_0(k,w,w_\tau) = \left[w_\tau^2 + \frac{(w^2+\nu^2k^4)^2}{4D_0^2}\right]^{-1} = \frac{1}{w_\tau}\,\mathrm{Im}\,G_0, \tag{25}$$

as required by the fluctuation dissipation theorem in the fictitious time. To get the correlation function of the original theory, we need to consider $C(k,w,\tau_1,\tau_2)$; set $\tau_1=\tau_2=\tau$ and let $\tau\to\infty$. The part which survives when $\tau\to\infty$ is obtained directly from the equal τ from Eq. (25), i.e., the result which is obtained by integrating the right-hand side of Eq. (25) over w_τ. This yields $2D_0(w^2+\nu^2k^4)^{-1}$ for the correlation function of the Edwards Wilkinson model, leading to the result $\alpha=(2-D)/2$.

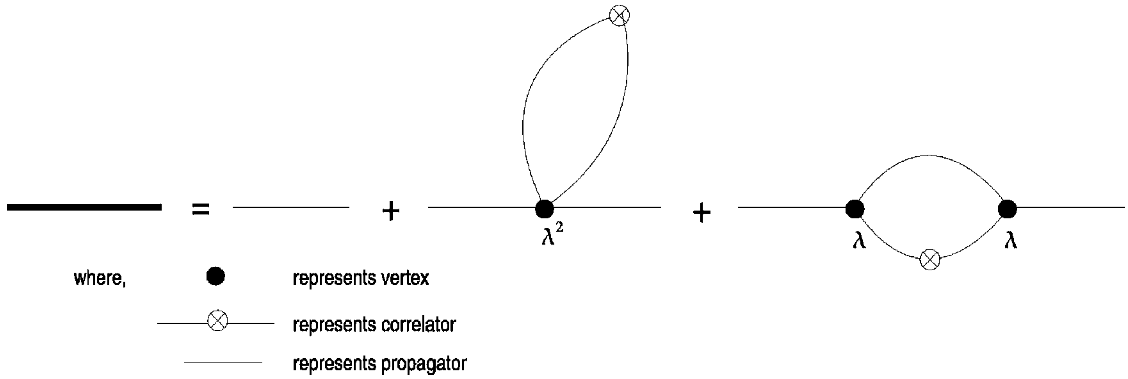

FIG. 1. Diagrams for the response function G.

With $\lambda \neq 0$, we develop as usual the fully dressed Greens function from perturbation theory and write

$$G = -iw_\tau + \frac{w^2 + \nu^2 k^4}{D_0} + \Sigma(k,w,w_\tau). \tag{26}$$

The point of our approach is that the $w=0$ part of Σ gives the flow ν and the $k=0$ part of Σ gives the flow of D. Further, mode coupling analysis will directly yield the exponent z from a power-counting analysis. As in other studies the result $\alpha+z=2$ holds in this case as well.

If we are interested in $\Sigma(k,w,w_\tau)$ to $O(\lambda^2)$, then we note that the contribution comes from two different sources—a one-loop contribution from the $O(\lambda^2)$ term in Eq. (23) and a two-loop contribution from the $O(\lambda)$ term. The contribution of the $O(\lambda^2)$ term can be read off from Eq. (23) by contracting two of the ϕ fields. The resulting correction to $(\Omega^2 + \nu^2 l^4)/2D_0$ does not have any new momentum dependence, and hence it is the second-order contribution from the $O(\lambda)$ term which is of significance. The expression for $\Sigma(l,\Omega,w_\tau)$ is found after standard alegbra as

$$\begin{aligned}\Sigma(l,\Omega,w_\tau) = \frac{\lambda^2}{8D_0^2}\int \frac{d^Dk}{(2\pi)^D}\frac{dw}{2\pi}\frac{dw_\tau}{2\pi}\{&[2(-iw+\nu k^2)\vec{l}\cdot(\vec{l}-\vec{k})\\ &-(-\Omega+\nu l^2)\vec{k}\cdot(\vec{l}-\vec{k})][2(-iw+i\Omega+\nu(\vec{l}\\ &-\vec{k})^2\vec{l}\cdot\vec{k})+(iw+\nu k^2)\vec{l}\cdot(\vec{l}-\vec{k})]G(k,w,w_\tau')\\ &\times C(\vec{l}-\vec{k},\Omega-w,w_\tau-w_\tau')\\ &+\text{similar term interchange of }\vec{k}\\ &\text{and }(\vec{l}-\vec{k})\text{ except in the first factor}\}.\end{aligned} \tag{27}$$

We now note that the Green's function can be written to $O(\lambda^2)$ as

$$G^{-1}(l,w,w_\tau) = -iw_\tau + \frac{1}{2}(\Omega^2/D_0 + \nu_{eff}^2 l^4/D), \tag{28}$$

where

$$\nu_{eff}^2 l^4/2D = \nu^2 l^4/2D_0 + \Sigma(l,\Omega,w_\tau). \tag{29}$$

If D_0 is not renormalized as happens for the KPZ system with colored noise, then one can carry out a power-counting argument at this stage. If however, D_0 is renormalized, as happens in this case, we need to expand ν_{eff} and D about ν and D_0 and, noting that the renormalization of ν dominates, we write

$$\nu_{eff} l^2 \simeq \nu l^2 + \frac{1}{2\nu l^2}\Sigma(l,\Omega,w_\tau) \tag{30}$$

or

$$\Delta\nu l^2 = \frac{1}{2\nu l^2}\Sigma(l,\Omega,w_\tau). \tag{31}$$

In a self-consistent mode coupling, we now replace ν by $\Delta\nu$ in Eq. (27), use G as given by Eq. (28), and C as follows from the fluctuation dissipation theorem. We can carry out the momentum count of Eq. (27), keeping in mind that $\nu l^2 \sim l^z$, to find

$$\Sigma \sim l^D l^z l^{2z+4} l^{-4z} = l^{D+4-z}. \tag{32}$$

Using this in Eq. (31), we have $l^z \sim l^{D+2-z}$, leading to

$$z = 1 + \frac{D}{2}. \tag{33}$$

Our analysis is not valid for $D>2$, because then Σ no longer dominates l^2. We have checked that, from Eq. (27) and Eq. (32), one can write flows for ν D_0 and, as in the usual analysis, there is a critical D, $1<D<2$, where the flow of the coupling constant $\lambda^2 D_0/\nu^3$ changes sign. We do not consider that to be of any particular significance one way or another. By introducing a fictitious time, we are able to bring back the fluctuation dissipation theorem, albeit in a fictitious time, and this allows a power-counting scaling analysis for the KPZ system. It is well known that the fluctuation dissipation theorem holds in $D=1$ for KPZ and $z=3/2$ is the exact answer. Our analysis, however, is not restricted to $D=1$. It is valid for $D<2$. Our approach is also useful for setting up a mean-field theory of turbulence, as we shall demonstrate elsewhere. It should be noted that a similar scheme, in a different context, has been promoted recently by Berges and Stamatescu [22]. They address the question of calculating correlation functions of the out-of-equilibrium quantum fields and show that the numerical procedure becomes efficient when one uses a technique based on stochastic quantization. This might have implications for glassy dynamics in statistical physics, which we intend to explore in the future.

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APPENDIX

The diagrams for the response function up to order λ^2 are shown in Fig. 1.

There is no k and w dependence in the second diagram of the right-hand side. The main contribution comes from the third diagram. Hence, the response function G up to order λ^2 is the following:

$$G(\vec{l},\Omega,w_\tau) = G_0(\vec{l},\Omega,w_\tau) + \frac{\lambda^2}{8D_0^2}G_0^2(\vec{l},\Omega,w_\tau)\Sigma(\vec{l},\Omega,w_\tau), \tag{A1}$$

where

$$\Sigma(\vec{l},\Omega,w_\tau) = \int \frac{d^Dk}{(2\pi)^D}\int \frac{dw}{(2\pi)}\int \frac{dw_\tau}{(2\pi)}\{[2(-iw+\nu k^2)\vec{l}\cdot(\vec{l}-\vec{k}) - (i\Omega+\nu l^2)\vec{k}\cdot(\vec{l}-\vec{k})][2(-i\Omega+\nu l^2)\vec{k}\cdot(\vec{k}-\vec{l}) - (iw+\nu k^2)\vec{l}\cdot(\vec{k} - \vec{l})]G_0(\vec{k},w,w_\tau)C_0(\vec{l}-\vec{k},\Omega-w,w_\tau-w'_\tau) + [2(-iw+\nu k^2)\vec{l}\cdot(\vec{l}-\vec{k}) - (i\Omega+\nu l^2)\vec{k}\cdot(\vec{l}-\vec{k})][2(-i\Omega+\nu l^2)(\vec{l}-\vec{k})\cdot(- \vec{k}) - \{i(\Omega-w)+\nu(\vec{l}-\vec{k})^2\}\vec{l}\cdot(-\vec{k})]G_0(\vec{l}-\vec{k},\Omega-w,w_\tau-w'_\tau)C_0(\vec{k},w,w_\tau)\}. \tag{A2}$$

Now, $\Sigma(\vec{l},\Omega,w_\tau)=\Sigma_{i=1}^{8}I_i$ We now calculate I_1 explicitly.

$$I_1 = \int \frac{d^Dk}{(2\pi)^D}\int \frac{dw}{(2\pi)}\int \frac{dw_\tau}{(2\pi)}4(-iw+\nu k^2)\vec{l}\cdot(\vec{l}-\vec{k})(-i\Omega+\nu l^2)\vec{k}\cdot(\vec{k}-\vec{l})G_0(\vec{k},w,w_\tau)C_0(\vec{l}-\vec{k},\Omega-w,w_\tau-w'_\tau). \tag{A3}$$

A change of variables $\vec{k}\to\vec{l}/2+\vec{k}$, $w'_\tau\to w_\tau/2+w'_\tau$, $w\to\Omega/2+w'$ leads to the symmetrized version of I_1, i.e.,

$$I_1 = 4\int \frac{d^Dk}{(2\pi)^D}\int \frac{dw}{(2\pi)}\int \frac{dw_\tau}{(2\pi)}[-i(\Omega/2+w) + \nu(\vec{l}/2+\vec{k})^2]\vec{l}\cdot(\vec{l}/2-\vec{k})(-i\Omega+\nu l^2)(\vec{l}/2+\vec{k})(\vec{k}-\vec{l}/2)\left[\frac{1}{i(w_\tau/2-w'_\tau)+\frac{1}{2D_0}[(\Omega/2-w)^2+\nu^2(\vec{l}/2-\vec{k})^4]}\right] \times\left[\frac{1}{-i(w_\tau/2+w'_\tau)+\frac{1}{2D_0}[(\Omega/2+w)^2+\nu^2(\vec{l}/2+\vec{k})^4]}\right]\left[\frac{1}{-i(w_\tau/2-w'_\tau)+\frac{1}{2D_0}[(\Omega/2-w)^2+\nu^2(\vec{l}/2-\vec{k})^4]}\right]. \tag{A4}$$

We are going to evaluate I_1 in the limit $\vec{l}\to 0$, $\Omega\to 0$ and $w_\tau\to 0$. Now, integrating over w'_τ and taking the limit $w_\tau\to 0$, we get

$$I_1 = 8D_0^2\int \frac{d^Dk}{(2\pi)^D}\int \frac{dw}{(2\pi)}\frac{[-i(\Omega/2+w)+\nu(\vec{l}/2+\vec{k})^2]\vec{l}\cdot(\vec{l}/2-\vec{k})(-i\Omega+\nu l^2)(\vec{l}/2+\vec{k})(\vec{k}-\vec{l}/2)}{[(\Omega/2-w)^2+\nu^2(\vec{l}/2-\vec{k})^4][2w^2+\Omega^2/2+\nu^2(\vec{l}/2-\vec{k})^4+\nu^2(\vec{l}/2+\vec{k})^4]}. \tag{A5}$$

After that, integrating over w and taking the limit $\Omega\to 0$, we get

$$\begin{aligned} I_1 &= \int \frac{d^Dk}{(2\pi)^D}\frac{\nu l^2\vec{l}\cdot(\vec{l}/2-\vec{k})(\vec{l}/2+\vec{k})\cdot(\vec{k}-\vec{l}/2)}{\nu^2(\vec{l}/2+\vec{k})^4}\left[\frac{(\vec{l}/2+\vec{k})^2+(\vec{l}/2-\vec{k})^2}{2(\vec{l}/2-\vec{k})^2} - \frac{(\vec{l}/2+\vec{k})^2+\sqrt{(\vec{l}/2-\vec{k})^4+(\vec{l}/2+\vec{k})^4}}{\sqrt{(\vec{l}/2-\vec{k})^4+(\vec{l}/2+\vec{k})^4}}\right] \\ &= \frac{8D_0^2}{\nu}\int dkk^{D-1}\frac{S_{D-1}}{(2\pi)^D}\int_0^\pi d\theta\sin^{D-2}\theta\frac{l^2[\vec{l}\cdot(\vec{l}/2-\vec{k})][(\vec{l}/2+\vec{k})\cdot(\vec{k}-\vec{l}/2)]}{(\vec{l}/2+\vec{k})^4} \\ &\quad -\left[\frac{(\vec{l}/2+\vec{k})^2+(\vec{l}/2-\vec{k})^2}{2(\vec{l}/2-\vec{k})^2}\frac{(\vec{l}/2+\vec{k})^2+\sqrt{(\vec{l}/2-\vec{k})^4+(\vec{l}/2+\vec{k})^4}}{\sqrt{(\vec{l}/2-\vec{k})^4+(\vec{l}/2+\vec{k})^4}}\right] = -\frac{8D_0^2}{\nu}A_\theta\int_0^\Lambda dkk^{D-3}, \end{aligned} \tag{A6}$$

<>where $A_\theta = \frac{S_D}{(2\pi)^D}\left[\frac{1}{2D} - \frac{1}{4} + \frac{1}{4\sqrt{2}}\right]$ and we have used $\vec{l}\cdot\vec{k} = lk\cos\theta$, $\frac{S_{D-1}}{(2\pi)^D}\int_0^\pi \sin^{D-2}\theta d\theta = S_D/(2\pi)^D$, and $\frac{S_{D-1}}{(2\pi)^D}\int_0^\pi \sin^{D-2}\theta \cos^2\theta d\theta = S_D/D(2\pi)^D$.

Now, from Eq. (A1), we get

$$\frac{1}{\nu_{eff}^2 l^4} = \frac{1}{\nu^2 l^4}\left[1 - \frac{\lambda^2 D_0}{\nu^3} 2A_\theta \int_0^\Lambda dk k^{D-3}\right] \tag{A7}$$

Similarly, we can evaluate all other I_i integrals. Finally, from Eq. (A2) we will get $\Sigma(\vec{l},\Omega,w_\tau)$. Then, by using Eq. (A1) we will get the same form of Eq. (A7) but with different A_θ.

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